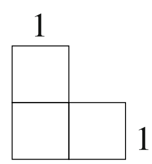
PUTNAM SEMINAR, SEPTEMBER 15 2016

Some basic proof strategies:

- Contradiction
- Induction
- Pigeonhole principle: If $kn + 1$ objects ($k \geq 1$ not necessarily finite) are distributed among n boxes, one of the boxes will contain at least $k + 1$ objects.
- Extremality: consider the "largest" (or "smallest") possible object/configuration.

Problem 1. Prove that for any $n \geq 1$, a $2^n \times 2^n$ checkerboard with one 1×1 corner square removed can be tiled by "L" shaped pieces as in the figure below



Problem 2. Given nine points inside the unit square, prove that some three of them form a triangle whose area does not exceed $1/8$.

Problem 3. Given n points in the plane, no three of which are collinear, show that there exists a closed polygonal line with no self-intersections having these points as vertices.

Problem 4. The vertices of a convex polygon are colored by at least three colors in such a way that no two consecutive vertices have the same color. Prove that one can dissect the polygon into triangles by diagonals that do not cross and whose endpoints have different colors.

Problem 5. A sequence of m positive integers contains exactly n distinct terms. Prove that if $2n \leq m$ then there exists a block of consecutive terms whose product is a perfect square.

Problem 6. In each of the unit squares of a 10×10 checkerboard, a positive integer not exceeding 10 is written. Any two numbers that appear in adjacent or diagonally adjacent squares of the board are relatively prime. Prove that some number appears at least 17 times.

Problem 7. Every point of three-dimensional space is colored red, green, or blue. Prove that one of the colors attains all distances, meaning that any positive real number represents the distance between two points of this color.

Problem 8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(\frac{x_1+x_2}{2}) = \frac{f(x_1)+f(x_2)}{2}$ for any $x_1, x_2 \in \mathbb{R}$. Show that

$$f\Big(\frac{x_1 + x_2 + \cdots + x_n}{n}\Big) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

for any $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

Problem 9. Consider a planar region of area 1, obtained as the union of finitely many disks. Prove that from these disks we can select some that are mutually disjoint and have total area at least 1/9.

Problem 10. The entries of a matrix are real numbers of absolute value less than or equal to 1, and the sum of the elements in each column is 0. Prove that we can permute the elements of each column in such a way that the sum of the elements in each row will have absolute value less than or equal to 2.