PUTNAM SEMINAR, SEPTEMBER 26 2019: ENUMERATIVE COMBINATORICS

Enumerative combinatorics deals with counting things, typically the number of ways that certain patterns can be formed. Some typical strategies to keep in mind:

- Sum rule
- Product rule
- Recursion
- Counting by bijection, or counting the same objects in two different ways
- Principle of inclusion and exclusion

Sample problem: Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. (Solution at the end of the worksheet).

Problem 1. How many words of length 5 from the alphabet $\{0, 1, 2, \ldots, 9\}$ have: (a) Strictly increasing digits; (b) Strictly increasing or strictly decreasing digits; (c) Nondecreasing digits; (d) Nondecreasing or nonincreasing digits.

Problem 2. $2n$ players are participating in a tennis tournament. Find the number of possible pairings for the first round.

Problem 3. $2n$ objects of each of three kinds are given to two persons, so that each person gets $3n$ objects. Prove that this can be done in $3n^2 + 3n + 1$ ways.

Problem 4.

(a) Find the number of diagonals of a convex n-gon.
(b) Find the number of intersection points of the diagonals of a convex n-gon.
(c) We draw all diagonals of a convex n-gon. Suppose no three diagonals pass through a point. Into how many parts is the n-gon divided?
(d) We draw all diagonals of a convex n-gon. Suppose no three diagonals pass through a point. Find the number of triangles.

Problem 5.

(a) Find a closed formula for $\sum_{k=1}^{n} \binom{n}{k} k^2$. Hint: interpret it as the number of ways of choosing a committee, its chairperson, and its secretary (possibly the same person) from a set with n persons.
(b) Find a closed formula for $\sum_{k=1}^{n} \binom{n}{k} k^3$.

Problem 6. Find combinatorial proofs of the following formulas (use bijection or counting the same objects in two ways):

(a) $\displaystyle\binom{n}{s} = \frac{n}{s}\binom{n-1}{s-1}$

(b) $\displaystyle\binom{n}{s} = \frac{n}{n-s}\binom{n-1}{s}$

(c) $\displaystyle\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$

(d) $\displaystyle\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}=\binom{2n}{n}=\sum_{k=0}^{n}\binom{n}{k}^2$

(e) $\displaystyle\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots$

(f) $\displaystyle\sum_{k=0}^{r}\binom{n+k}{k}=\binom{n+r+1}{r}$

Problem 7. Let $f(n)$ be the number of words of length n without neighboring zeros from the alphabet $\{0,1,2\}$. Find a recursion and a closed formula for $f(n)$. (For the closed formula, you may need to take a look at the next problem).

Problem 8. (Closed formulas for recurrence equations) You may be familiar with the famous Fibonacci sequence: it is given by $x_0 = 1, x_1 = 1$ and then continued via the rule $x_n = x_{n-1} + x_{n-2}$. More generally, let's consider a sequence given by the recurrence equation $x_n = px_{n-1} + qx_{n-2}$.

(a) Assuming that the solution has the form $x_n = \lambda^n$, show that the number λ should satisfy $\lambda^2 - p\lambda - q = 0$.
(b) If λ_1, λ_2 satisfy the quadratic equation above, show that $x_n = a\lambda_1^n + b\lambda_2^n$ satisfies the recurrence equation.
(c) Find a formula for the n-th term of the Fibonacci sequence.

Problem 9. Consider all $2^n - 1$ nonempty subsets of the set $\{1, 2, \ldots, n\}$. For every such subset we find the product of the reciprocals of each of its elements. Find the sum of all these products.

Problem 10. In how many ways can you select two disjoint subsets from a set with n elements?

Problem 11. In how many ways can you take an odd number of objects from n objects?

Problem 12. How many subsets of $\{1, 2, \ldots, n\}$ have no two successive numbers?

Problem 13. The number of binary words of length n with exactly m 01-blocks is $\binom{n+1}{2m+1}$.

Problem 14. Prove that $\sum_{k=0}^{n}\binom{n+k}{k}\frac{1}{2^k} = 2^n$.

Problem 15. How many words of length n from the alphabet $\{0,1,2\}$ are such that neighbors differ at most by 1?

Problem 16. Does a polyhedron exist with an odd number of faces, each face having an odd number of edges?

Solution to sample problem: The sum on the left counts the number of ways we can choose a subset from a set with n elements: the number of subsets with zero elements, plus the number of subsets with 1 element, and so on. Let us now observe that the total number of subsets is precisely the number of binary words of length n. Place the n elements in a row, and each binary word encodes a subset: having a 1 in the m-th position means that we select the m-th element to be part of the subset. Clearly, by the product rule, there are 2^n such binary words, and the formula is proved.