

PUTNAM SEMINAR, OCTOBER 3 2019: NUMBER THEORY

Today's theme is problems related to Number Theory.

Problem 1. How many zeros are at the end of $1000!$?

Problem 2. $641 \mid 2^{32} + 1$. No calculators allowed!

Problem 3. $2^n \nmid n!$

Problem 4. Among five integers, there are always three with sum divisible by 3.

Problem 5. $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1 = 0$ with $a_i \in \{1, -1\}$. Show that $4 \mid n$.

Problem 6. Three brothers inherit n gold pieces weighting $1, 2, \dots, n$. For what n can they be split into three equal heaps?

Problem 7. Find the five-digit number $abcde$ such that $4 \cdot abcde = edcba$.

Problem 8. The sequence $a_n = \sqrt{24n + 1}$, $n \in \mathbb{N}$, contains all primes except 2 and 3.

Problem 9. $m \mid (m - 1)! + 1 \Rightarrow m$ is a prime.

Problem 10. Find the smallest positive integer with the property that, if you move the first digit to the end, the new number is 1.5 times larger than the old one.

Problem 11. A 20-digit positive integer starting with 11 ones cannot be a square.

Problem 12. $x^2 + y^2 = x^2y^2$ has no integral solutions besides $x = y = 0$.

Problem 13. Find all integral solutions of $x^2 + y^2 + z^2 = x^2y^2$.

Problem 14. Find all integral solutions of $x + y = x^2 - xy + y^2$.

Problem 15. Different lattice points of the plane have different distances from $(\sqrt{2}, \frac{1}{3})$.

Problem 16. Find pairwise prime solutions of $1/x + 1/y = 1/z$.

Problem 17. Find pairwise prime solutions of $1/x^2 + 1/y^2 = 1/z^2$.

Problem 18. The next to last digit of 3^n is even.

Problem 19. There are black and white balls in an urn. If you draw two balls at random, the probability is $1/2$ to get a mixed couple. What can you conclude about the contents of the urn?

Problem 20. If you are condemned to die in Sikinia, you are put into Death Row until the last day of the year. Then all prisoners from Death Row are arranged in a circle and numbered $1, 2, \dots, n$. Starting with #2 every second one is shot until only one remains who is immediately set free. How do you find the place of the sole survivor?