PUTNAM SEMINAR, OCTOBER 10 2019: INEQUALITIES

Today's theme is problems involving inequalities.

Problem 1. The positive integers a, b, c are such that $a^2+b^2-ab = c^2$. Prove that $(a-c)(b-c) \le 0$.

Problem 2. If a, b, c are real numbers such that $0 \le a, b, c \le 1$, then

$$\frac{a}{1+bc} + \frac{b}{1+ac} + \frac{c}{1+ab} \le 2.$$

Problem 3. Let a, b, c be the side lengths of a triangle, and let s_a, s_b, s_c be the lengths of the medians. D is the diameter of the circumcircle. Prove that

$$\frac{a^2+b^2}{s_c} + \frac{a^2+c^2}{s_b} + \frac{b^2+c^2}{s_a} \le 6D.$$

Problem 4. Each of the vectors $\vec{a}_1, \dots, \vec{a}_n$ has length ≤ 1. Prove that the signs can be chosen in the sum

$$\vec{c} = \pm\vec{a}_1 \pm \cdots \pm \vec{a}_n \qquad \text{so that} \qquad |\vec{c}| \le \sqrt{2}.$$

Problem 5. Fifty watches, all showing correct time, are on a table. Prove that at a certain moment the sum of the distances from the center O of the table to the endpoints of the minute hands is greater than the sums of the distances from O to the centers of the watches.

Problem 6. Which of the two numbers is larger:

(1) An exponential tower of n 2's or an exponential tower of $(n-1)$ 3's?
(2) An exponential tower of n 3's or an exponential tower of $(n-1)$ 4's?

Problem 7. The vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ with sum $\vec{0}$ are given in a plane. Prove the inequality

$$|\vec{a}| + |\vec{b}| + |\vec{c}| + |\vec{d}| \ge |\vec{a}+\vec{d}| + |\vec{b}+\vec{d}| + |\vec{c}+\vec{d}|$$

Prove this also for one and three dimensions.

Problem 8. The Fibonacci sequence is defined by $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$. Prove that

$$\frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{5}{2^5} + \cdots + \frac{a_n}{2^n} < 2.$$

Problem 9. Prove that

$$\frac{1}{15} < \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdots \frac{99}{100} < \frac{1}{10}.$$

Problem 10. Let $\{a_k\}$ be a sequence of pairwise distinct positive integers. Show that for all positive integers n

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \ \ge \ \sum_{k=1}^{n} \frac{1}{k}.$$

Problem 11. A point is chosen on each side of a unit square. The four points are sides of a quadrilateral with sides a, b, c, d. Show that

$$2 \leq a^2 + b^2 + c^2 + d^2 \leq 4 \qquad \text{and} \qquad 2\sqrt{2} \leq a + b + c + d \leq 4.$$

Problem 12. Let $0 < a \leq b \leq c \leq d$. Then $a^b b^c c^d d^a \geq b^a c^b d^c a^d$.

Problem 13. For $x, y, z > 0$,

$$\text{(a)} \quad \frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{x}{y} + \frac{y}{z} + \frac{z}{x}, \qquad \text{(b)} \quad \frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{y}{x} + \frac{z}{y} + \frac{x}{z}.$$

Problem 14. The polynomial $ax^2 + bx + c$ with $a > 0$ has real roots x_1, x_2. Prove that $|x_i| \leq 1$ $(i = 1, 2)$ exactly if $a + b + c \geq 0$, $a - b + c \geq 0$, $a - c \geq 0$.

Problem 15. The following inequality holds for any triangle with sides a, b, c:

$$a\,(b^2 + c^2 - a^2) + b\,(a^2 + c^2 - b^2) + c\,(a^2 + b^2 - c^2) \leq 3abc.$$

Problem 16. The following inequality holds for any triangle with sides a, b, c:

$$a^2\,b\,(a - b) + b^2\,c\,(b - c) + c^2\,a\,(c - a) \geq 0.$$