PUTNAM SEMINAR, OCTOBER 17 2019: THE INDUCTION PRINCIPLE

Recall that induction is a proof technique used to prove that a property $P(n)$ holds for every natural number n, i.e. for $n = 1, 2, 3, \ldots$ It consists of two parts:

- The **base case**: proving the property for $n = 1$.
- The **induction step**: assuming that we already know the property holds for $n = k$, we prove it for $n = k + 1$.

There are slight variations of this technique, the most common one being *strong induction*: for the step, we assume that the property holds for all $n \le k$ and then prove it for $n = k + 1$.

Problem 1. Define the Fibonacci sequence by $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$ for $n \ge 0$. The following formulas involving Fibonacci numbers can be proved by induction:

(a) $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, where $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$.
(b) $F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots$
(c) $\sum_{j=1}^{n} F_j^2 = F_n F_{n+1}$.
(d) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$
(e) $F_{n-1}F_{n+1} = F_n^2 + (-1)^n$.
(f) $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$.
(g) $m|n \Rightarrow F_m|F_n$.
(h) $\gcd(F_m, F_n) = F_{\gcd(m,n)}$.
(i) Define

$$t_1 = 1, \quad t_2 = 1 + \frac{1}{1}, \quad t_3 = 1 + \frac{1}{1 + \frac{1}{1}}, \quad t_4 = 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1}}}, \cdots$$

Prove that $t_n = F_{n+1}/F_n$.

Problem 2. n circles are given in the plane. They divide the plane into parts. Show that you can color the plane with two colors, so that no parts with a common boundary line are colored the same way. Such a coloring is called a proper coloring.

Problem 3. There are n identical cars on a circular track. Among all of them, they have just enough gas for one car to complete a lap. Show that there is a car which can complete a lap by collecting gas from the other cars on its way around.

Problem 4. Every road in Sikinia is one-way. Every pair of cities is connected by exactly one direct road. Show that there exists a city which can be reached from every other city either directly or via at most one other city.

Problem 5. Let α be a real number such that $\alpha + \frac{1}{\alpha}$ is an integer. Prove that $\alpha^n + \frac{1}{\alpha^n}$ is an integer for any natural number n.

Problem 6. All numbers of the form $12008, 120308, 1203308, \ldots$ are divisible by 19.

Problem 7. Show that $\sum_{k=0}^{n} \binom{n+k}{k} \frac{1}{2^k} = 2^n$.

Problem 8. Define $a_0 = 1$, and $a_{n+1} = \sqrt{2}^{a_n}$. Show that the sequence a_n is nondecreasing and bounded above by 2.

Problem 9. Let a_n be the number of words of length n from the alphabet $\{0, 1\}$, which do not have two 1's at distance 2 apart. Find a_n in terms of the Fibonacci numbers.

Problem 10. We are given N lines ($N > 1$) in a plane, no two of which are parallel and no three of which have a point in common. Prove that it is possible to assign a non-zero integer of absolute value not exceeding N to each region of the plane determined by these lines, such that the sum of the integers on either side of any of the given lines is equal to 0.

Problem 11. Suppose that $z + 1/z = 2\cos\alpha$. Show that $z^n + 1/z^n = 2\cos(n\alpha)$ for all $n \in \mathbb{N}$.

Problem 12. For any integer $n \geq 0$, $3^{n+1} | 2^{3^n} + 1$.

Problem 13. Let $P_1, \ldots, P_{2n+1}$ be points on the unit circle centered at the origin, all on the same side of a diameter. Prove that

$$|\overrightarrow{OP}_1 + \cdots + \overrightarrow{OP}_{2n+1}| \geq 1.$$

Problem 14. Consider all possible subsets of the set $\{1, 2, ..., N\}$, which do not contain any neighboring elements. Prove that the sum of the squares of the products of all numbers in these subsets is $(N + 1)! - 1$.

Problem 15. In an $m \times n$ matrix of real numbers, we mark at least p of the largest numbers ($p \leq m$) in every column, and at least q of the largest numbers ($q \leq n$) in every row. Prove that at least pq numbers are marked twice.

Problem 16. n points are selected along a circle and labeled a or b. Prove that there are at most $\lfloor \frac{3n+4}{2} \rfloor$ chords which join differently labeled points and which do not intersect inside the circle.

Problem 17. Let $n = 2^k$. Prove that we can select n integers from any $(2n - 1)$ integers such that their sum is divisible by n.

Problem 18. For any natural number n, prove that inequality

$$\sqrt{2\sqrt{3\sqrt{4\cdots\sqrt{(n-1)\sqrt{n}}}}} < 3$$

Problem 19. Let $a_1 \leq a_2 \leq \cdots \leq a_n$ be positive integers such that $1/a_1 + 1/a_2 + \cdots + 1/a_n = 1$. Prove that $a_n < 2^{n!}$.

Problem 20. $2n$ points are given in space, and $n^2 + 1$ line segments are drawn between pairs of these points. Show that there is at least one set of three points which are joined pairwise by line segments.