# PUTNAM SEMINAR, OCTOBER 24 2019: SEQUENCES

Today's theme is problems involving sequences.

**Problem 1.** A sequence is given by

$$a_0 = 2, \qquad a_1 = 7, \qquad a_{n+1} = 7a_n - 12a_{n-1}.$$

Find a closed expression for $a_n$.

**Problem 2.** $a_1 = a_2 = 1,\ a_n = (a_{n-1}^2 + 2)/a_{n-2},\ (n \geq 3)$. Show that all $a_i$ are integers.

**Problem 3.** $a_1 = a,\ a_2 = b,\ a_{n+2} = (a_{n+1} + a_n)/2,\ (n \geq 0)$. Find $\lim_{n\to\infty} a_n$.

**Problem 4.** Let $a_n = \frac{2^3-1}{2^3+1} \cdot \frac{3^3-1}{3^3+1} \cdot \frac{4^3-1}{4^3+1} \cdot \ldots \cdot \frac{n^3-1}{n^3+1}$. Find $\lim_{n\to\infty} a_n$.

**Problem 5.** All terms of the sequence $10001, 100010001, 1000100010001, \ldots$ are composite.

**Problem 6.** Given a set of positive numbers, the sum of the pairwise products of its elements is equal to 1. Show that it is possible to eliminate one number so that the sum of the remaining numbers is less than $\sqrt{2}$.

**Problem 7.** In how many ways can you tile a $4 \times n$ rectangle by $3 \times 1$ dominoes?

**Problem 8.** In how many ways can you fill a $2 \times 2 \times n$ box with $1 \times 1 \times 2$ bricks? A table suggests that the values $a_{2n}$ are squares. Can you prove this?

**Problem 9.** Does there exist a positive sequence $a_n$ such that $\sum a_n$ and $\sum 1/(n^2 a_n)$ are convergent?

**Problem 10.** Investigate the sequence

$$a_n = \binom{n}{0}^{-1} + \binom{n}{1}^{-1} + \cdots + \binom{n}{n}^{-1}.$$

Is it bounded? Does it converge for $n \to \infty$?

**Problem 11.** If $r > 0$ is a rational approximation of $\sqrt{5}$, then $(2r+5)/(r+2)$ is an even better approximation. Generalize to $\sqrt{a}$.

**Problem 12.** A sequence is defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \left(\sqrt{2}\right)^{a_n}$. Find $\lim_{n\to\infty} a_n$.

**Problem 13.** *Morse-Thue Sequence.* Start with 0; to each initial segment append its complement:

$$0, \quad 01, \quad 0110, \quad 01101001, \quad 0110100110010110, \quad \ldots$$

Let the digits of the sequence be $x(0), x(1), x(2), \ldots$.

(1) Prove that $x(2n) = x(n)$ and $x(2n+1) = 1 - x(2n)$.
(2) Prove that $x(n) = 1 - x(n - 2^k)$ where $2^k$ is the largest power of 2 which is $\leq n$.
(3) Find the 1993rd digit of the sequence.
(4) Prove that the sequence is not periodic.
(5) Write the nonnegative integers in base 2: $0, 1, 10, 11, \ldots$. Now replace each number by the sum of its digits mod 2. You get the Morse-Thue sequence. Prove this.

**Problem 14.** *The arithmetic-geometric mean of Gauss.* Let $0 < a < b$. We define two sequences $a_n$ and $b_n$ as follows.

$$a_0 = a, \qquad b_0 = b, \qquad a_{n+1} = \sqrt{a_n b_n}, \qquad b_{n+1} = \frac{a_n + b_n}{2}.$$

(1) Prove that $a_n < a_{n+1}$, $b_n > b_{n+1}$, and $a_n < b_n$ for all $n$.
(2) Prove that $b_{n+1} - a_{n+1} = (b_n - a_n)^2/8b_{n+2}$.
(3) Show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = g$ with a quadratic convergence rate.

**Problem 15.** Find a recursion for the number $a_n$ of permutations $p$ of $\{1, \ldots, n\}$ with $|p(i) - i| \leq 2$ for all $i$.

**Problem 16.** Find the sum $S_n = 1/1 \cdot 2 \cdot 3 \cdot 4 + \cdots + 1/n(n+1)(n+2)(n+3)$.