PUTNAM SEMINAR, NOVEMBER 7 2019: POLYNOMIALS

Today's theme is problems involving polynomials.

Problem 1. The polynomial $1 + x/1 + x^2/2! + \cdots + x^n/n!$ has no multiple roots.

Problem 2. Find a cubic equation whose roots are the third powers of the roots of $x^3 + ax^2 + bx + c$.

Problem 3. Let $n > 1$ be an integer. Show that $f(x) = x^n + 5x^{n-1} + 3$ is irreducible over $\mathbb{Z}$.

Problem 4. Show that $k|n \Leftrightarrow x^k - a^k | x^n - a^n$, where $a, k, x, n \in \mathbb{N}$.

Problem 5. If a and b are two solutions of $x^4 + x^3 - 1 = 0$, then ab is a solution of

$$x^6 + x^4 + x^3 - x^2 - 1 = 0$$

Problem 6. A polynomial $f(x) = x^4 + *x^3 + *x^2 + *x + 1$ has three undetermined coefficients denoted by stars. The players A and B move alternately, replacing a star by a real number until all stars are replaced. A wins if all zeros of the polynomial are complex. B wins if at least one zero is real. Show that B can win in spite of his only second move.

Problem 7. Find all pairs of positive integers (m, n) so that

$$1 + x + x^2 + \cdots + x^m \mid 1 + x^n + x^{2n} + \cdots + x^{mn}$$

Problem 8. Find all polynomial solutions of the functional equation

$$f(x)f(x+1) = f(x^2 + x + 1)$$

Problem 9. Let n be a positive integer, and let

$$f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}.$$

Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$.