**PUTNAM SEMINAR, NOVEMBER 13 2019:
FUNCTIONAL EQUATIONS**

An equation where the unknown in a function are known as *functional equations*. There are a few standard ideas you should apply when dealing with such an equation:

- Plug in special values: 0, 1, etc.
- If there are two variables: set them equal to each other, or one to be the negative of the other, etc.
- Change of variable: from x to $-x$, or some other transformation suggested by the equation.
- Take derivatives.

Problem 1. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(xy) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

Problem 2. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Repeat for monotonically increasing functions.

Problem 3. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

Problem 4. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\Big(\frac{x+y}{2}\Big) = \frac{f(x) + f(y)}{2}$$

for all $x, y \in \mathbb{R}$.

Problem 5. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y) + f(x - y) = 2f(x)f(y)$$

for all $x, y \in \mathbb{R}$.

Problem 6. Find some functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = f(x/2)$ for all $x \in \mathbb{R}$. Can you find them all?

Problem 7. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) + f(x - y) = 2f(x)\cos y$ for all $x, y \in \mathbb{R}$.

Problem 8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and suppose that for a fixed number $a \in \mathbb{R}$ we have that

$$f(x + a) = \frac{1 + f(x)}{1 - f(x)}$$

for all $x \in \mathbb{R}$. Prove that f is periodic.

Problem 9. Find all polynomials p satisfying $p(x + 1) = p(x) + 2x + 1$ for all $x \in \mathbb{R}$.

Problem 10. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$xf(y) + yf(x) = (x + 2)f(x)f(y)$$

for all $x, y \in \mathbb{R}$.

Problem 11. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)f(y) = f(x - y)$$

for all $x, y \in \mathbb{R}$. Can you find any discontinuous ones?

Problem 12. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y) + f(x - y) = 2\big[f(x) + f(y)\big]$$

for all $x, y \in \mathbb{R}$.

Problem 13. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y) - f(x - y) = 2f(y)$$

for all $x, y \in \mathbb{R}$.

Problem 14. Let $f : \mathbb{N} \to \mathbb{N}$ satisfy that $f(n + 1) > f(f(n))$ for each $n \in \mathbb{N}$. Show that $f(n) = n$ for all $n \in \mathbb{N}$.

Problem 15. Find all functions f defined on the set of positive real numbers, which take positive real values, and satisfy the conditions:
(a) $f\big(xf(y)\big) = yf(x)$ for all $x, y > 0$.
(b) $f(x) \to 0$ as $x \to \infty$.

Problem 16. Find all functions f defined on the set of positive rational numbers, which take positive rational values, and satisfy $f\big(xf(y)\big) = f(x)/y$ for all positive rationals x, y.

Problem 17. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x) + f(y) + xy$ for all $x, y \in \mathbb{R}$.

Problem 18. Find all functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ such that $xf(x) + 2xf(-x) = -1$ for all $x \in \mathbb{R} \setminus \{0\}$.

Problem 19. Find all polynomials over $\mathbb{C}$ satisfying $f(x)f(-x) = f(x^2)$ for all $x \in \mathbb{C}$.

Problem 20. Let $f : (0, \infty) \to \mathbb{R}$ be a strictly increasing function satisfying, for all $x > 0$,

$$f(x) > -\frac{1}{x}, \qquad f(x) \cdot f\big(f(x) + 1/x\big) = 1.$$

Find $f(1)$. Can you give a specific example of such a function?

Problem 21. Find all functions$f : \mathbb{N} \to \mathbb{N}$ satisfying $f(n) + f(f(n)) + f(f(f(n))) = 3n$ for each $n \in \mathbb{N}$.