PUTNAM SEMINAR, NOVEMBER 20 2019: GEOMETRY WITH VECTORS

Some geometry problems can be solved by using the algebraic tool of vectors. We identify each point in the problem with a vector in the plane (or 3D-space), and use the following important facts. Capital letters will denote vectors/points in the plane/space.

- Multiplication by a real number corresponds to stretching all vectors by that factor.
- $\overrightarrow{AB} = B - A$.
- The midpoint of the segment between A and B is $(A + B)/2$.
- The line through A and B can be described as the set of points of the form $tA + (1 - t)B$, where t is a real number. If we restrict t to the interval $[0, 1]$, we get the segment between A and B.
- Two vectors are perpendicular if and only if $A \cdot B = 0$. The square of the length of a vector A is $A \cdot A$.

Sample problem: The three medians of a triangle all pass through the same point (the centroid or center of mass).

Let A, B and C be the vertices of the triangle. The midpoint of BC is $\frac{1}{2}B + \frac{1}{2}C$. The median through A has then the equation $(1 - t)A + \frac{t}{2}B + \frac{t}{2}C$ with t in $[0, 1]$. In particular, the choice $t = 2/3$ shows that the point $\frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$ is on the median through A. Permuting the roles of A, B and C shows that this point is on all three medians. NOTE: This also shows that the centroid divides each median in a $2 : 1$ ratio. Why?

Problem 1. $ABCD$ is a parallelogram if and only if $A + C = B + D$.

Problem 2. The midpoints P, Q, R, S of any quadrilateral in the plane or in space are the vertices of a parallelogram.

Problem 3. Let $ABCDEF$ be any hexagon, and let $A_1B_1C_1D_1E_1F_1$ be the hexagon of the centroids of the triangles ABC, BCD, CDE, DEF, EFA, FAB. Show that the hexagon $A_1B_1C_1D_1E_1F_1$ has parallel and equal opposite sides.

Problem 4. The diagonals of a quadrilateral are orthogonal if and only if the sums of the squares of opposite sides are equal.

Problem 5. A median of a triangle connects a vertex with the midpoint of the opposite side. A median of a quadrilateral connects the midpoints of two opposite sides. Show that the diagonals of a quadrilateral are orthogonal if and only if its medians have equal length.

Problem 6. Let A, B, C, D be four points in space. Prove the following: if for all points X in space we have $|AX|^2 + |CX|^2 = |BX|^2 + |DX|^2$, then $ABCD$ is a rectangle.

Problem 7. The points E, F, G, H divide the sides of the quadrilateral $ABCD$ in the same ratios. Find the conditions for $EFGH$ to be a parallelogram.

Problem 8. Let Q be an arbitrary point in the plane and let M be the midpoint of AB. Then $|QA|^2 + |QB|^2 = 2|QM|^2 + |AB|^2/2$.

Problem 9. Let A, B, C, D be four points in space. Show that $|AC|^2 + |BD|^2 + |AD|^2 + |BC|^2 \geq |AB|^2 + |CD|^2$.

Problem 10. Let $\vec{v}_1, \ldots, \vec{v}_n$ be vectors in the plane of length at most 1. Show that we can choose signs so that the vector $\pm\vec{v}_1 \pm \cdots \pm \vec{v}_n$ has norm at most $\sqrt{2}$.

Problem 11. Let ABC be a triangle, and O any point in space. Show that $|AB|^2 + |BC|^2 + |CA|^2 \leq 3\big(|OA|^2 + |OB|^2 + |OC|^2\big)$. When does equality happen?

If you know a little bit about complex numbers, we can also use those to solve geometric problems. The important things to keep in mind are:

- Multiplication by i is a counterclockwise rotation by $90°$. More generally, multiplication by $e^{i\theta}$ is a counterclockwise rotation by θ.
- The n-th roots of unity form a regular n-gon.

Problem 12. If one constructs equilateral triangles outwardly (inwardly) on the sides of a triangle, then their centers form another equilateral triangle.

Problem 13. Squares are constructed outwardly on the sides of a quadrilateral. If the centers of the squares are x, y, z, u, then the segments xz and yu are perpendicular and of equal length.

Problem 14. Let A, B, C, D be four points in a plane. Then

$$|AB| \times |CD| + |BC| \times |AD| \geq |AC| \times |BD|.$$

Problem 15. OAB and $OA'B'$ are equilateral triangles of the same orientation, S is the centroid of OAB, and M and N are the midpoints of $A'B$ and AB', respectively. Show that the triangles SMB' and SNA' are similar.

Problem 16. Equilateral triangles with the vertices E, F, G, H are constructed on the sides of a plane quadrilateral $ABCD$. Let M, N, P, Q be the midpoints of the segments EG, HF, AC, BD, respectively. What is the shape of $PMQN$?

Problem 17. Equilateral triangles OAB, OA_1B_1 and OA_2B_2 are positively oriented with common vertex O. Show that the midpoints of BA_1, B_1A_2, and B_2A are vertices of an equilateral triangle.

Problem 18. Construct regular hexagons on the sides of a centrally symmetric hexagon. Their centers form the vertices of a regular hexagon.

Problem 19. Let $A_0, A_1, \ldots, A_{n-1}$ be the vertices of a regular n-gon inscribed in the unit circle. Prove that

$$|A_0A_1| \cdot |A_0A_2| \cdots |A_0A_{n-1}| = n.$$